**B. Math. (Hons.) IIIrd year**
**Second Semestral examination 2021**
**Algebraic Number Theory**
**Instructor: B. Sury**
**May 19, 2021; 10 AM - 1 PM.**
**Answer SIX questions INCLUDING Question 1.**
**Be BRIEF!**

**Q 1.** (Compulsory)
Let $K$ be a number field.
(i) Show that every non-zero ideal of $O_K$ contains a non-zero integer.
(ii) Show that every non-zero prime ideal of $O_K$ contains a unique prime number.
(iii) Show that there exists $\alpha \in O_K$ such that $K = \mathbb{Q}(\alpha)$.
(iv) Prove that $\frac{1+10^{1/3}+10^{2/3}}{3}$ is an algebraic integer.
(v) Show that the ring of all algebraic integers is not Noetherian.

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**Q 2.** Let $K = \mathbb{Q}(\sqrt{7}, \sqrt{10})$. Prove that $O_K \neq \mathbb{Z}[\alpha]$ for any $\alpha \in O_K$.

**OR**

If $A$ is an integral domain for which every non-zero fractional ideal is invertible. Prove that $A$ is Noetherian and integrally closed in its quotient field.

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**Q 3.** Let $K, L$ be algebraic number fields of degrees $m, n$ over $\mathbb{Q}$. Assume $[KL : \mathbb{Q}] = [K : \mathbb{Q}][L : \mathbb{Q}]$ and that disc $O_K$ and disc $O_L$ are relatively prime. Prove that disc $O_{KL} = (\text{disc } O_K)^{[L:\mathbb{Q}]}\ (\text{disc } O_L)^{[K:\mathbb{Q}]}$.
*Hint.* The hypothesis implies that $O_{KL} = O_K O_L$ - you may assume this.

**OR**

Show that a Dedekind domain with only finitely many prime ideals is a PID.

**Q 4.** Let $\alpha$ be an algebraic integer whose minimal polynomial is Eisensteinian for a prime $p$. Prove that $p$ does not divide $[O_K : \mathbb{Z}[\alpha]]$ where $K = \mathbb{Q}(\alpha)$.

**OR**

Let $A$ be an integrally closed domain. Let $f \in A[X]$ be a monic polynomial so that $f = gh$ where $g, h \in K[X]$ are monic with $K$, the quotient field of $A$. Prove that $g, h \in A[X]$.

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**Q 5.** Consider $L := \mathbf{Q}(\zeta_{p^2})$, where $\zeta_{p^2}$ is a primitive $p^2$-th root of unity with $p$ an odd prime. Let $K$ be the unique subfield of $L$ which has degree $p$ over $\mathbf{Q}$. Show that 2 splits completely in $K$ if and only if $2^{p-1} \equiv 1 \bmod p^2$.

**OR**

Let $p \equiv 1 \bmod 3$ be a prime. Let $K \subset \mathbb{Q}(\zeta_p)$ be the unique subfield whose degree is 3 over $\mathbb{Q}$. Prove that a prime $q \neq p$ splits completely in $O_K$ if and only if $q$ is a cube mod $p$.

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**Q 6.** Let $p$ be a prime of the form $2^n + 1$. If $K$ is the cyclotomic field generated by the $p$-th roots of unity, determine the number of prime ideals lying over 2 in terms of $n$.
*Hint:* You may use without proof the Kummer-Dedekind criterion.

**OR**

If $K$ is a cubic extension of $\mathbb{Q}$ with discriminant $-d$ where $0 < d < 50$, prove that $O_K$ is a PID.

**Q 7.** Let $K = \mathbb{Q}(\alpha)$ where $\alpha$ is the real number $19^{1/3}$. Then $O_K = \{\frac{a+b\alpha+c\alpha^2}{3} : a \equiv b \equiv c \ (mod\ 3)\}$ and the ideal $3O_K = P^2Q$ for some prime ideals $P \neq Q$ (assume this). Prove that the class number of $K$ is a multiple of 3.

**OR**

Let $p \neq q$ be distinct primes congruent to 1 modulo 4. Assume that $p$ is NOT a square modulo $q$. Then, prove that the class number of $\mathbb{Q}(\sqrt{pq})$ is even.

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**Q 8.** Determine the fundamental unit of $\mathbb{Q}(3^{1/3})$.

**OR**

If $p \equiv 1 \bmod 4$ is a prime, show that there are integers $a, b$ satisfying $a^2 - pb^2 = -1$.

**OR**

Let $d$ be a square-free integer such that for some $n$, the integer $\frac{4^n+1}{d}$ is a perfect square; say $u^2$. Prove that $2^n + u\sqrt{d}$ is the fundamental unit of $\mathbb{Q}(\sqrt{d})$ if $\mathbb{Q}(\sqrt{d}) \neq \mathbb{Q}(\sqrt{5})$.